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Generalized dromion solutions of the $(2+1)$-dimensional KdV equation

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Abstract. The rich dromion structures for a $(2+1)$-dimensional KdV equation are revealed. The dromions in a high dimensional integrable model may have a free shape in one or more directions. Multi-dromion solutions can be driven by perpendicular line, non-perpendicular line and curved line ghost solitons.

Recently, much progress in understanding the properties of high-dimensional integrable models has been achieved [1]. One of the most important properties is that exponentially localized structures, called dromions, which are driven by two perpendicular line ghost solitons in the case of the Davey–Stewartson (DS) equation [2] or two non-perpendicular line ghost solitons in the case of the Kadomtsev–Petviashvili (KP) equation [3], have been found. On the other hand, we also know that for higher dimensional integrable models, many (or even infinitely many) arbitrary functions can be included in their symmetry structures [4–6]. This means some arbitrary functions can be included in the exact solutions of the higher dimensional integrable models. In this paper, we would like to study the dromion structure in more generalized form for the following $(2+1)$-dimensional KdV equation

$$u_t + u_{xxx} = 3(u\partial_y^{-1}u_x)_x. \tag{1}$$

Equation (1) was originally derived by using the idea of the weak Lax pair [7, 8] and it reduces to the usual $(1+1)$-dimensional KdV equation in the case of $y = x$. In [8], the authors pointed out that the solutions of equation (1) and its potential

$$v = \partial_y^{-1}u_x \tag{2}$$

can all be obtained from the bilinear form

$$(D_y D_t + D_x^3 D_y)\phi \cdot \phi = 0 \tag{3}$$

with

$$u = -2\partial_x\partial_y \ln\phi \qquad v = -2\partial_x\partial_x \ln\phi \tag{4}$$

where D is the standard Hirota bilinear operator [9].

In order to get solutions of (3), we expand ϕ in the form of a power series of a small parameter

$$\phi = 1 + \epsilon\phi^{(1)} + \epsilon^{(2)}\phi^{(2)} + \cdots. \tag{5}$$

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Substituting equation (5) into (3) yields a set of linear equations:

$$\phi^{(1)}_{yt} + \phi^{(1)}_{xxxy} = 0 \tag{6}$$

$$\phi^{(2)}_{yt} + \phi^{(2)}_{xxxy} = -\tfrac{1}{2}(D_yD_t + D_x^3D_y)\phi^{(1)} \cdot \phi^{(1)} \tag{7}$$

etc. In [8], Radha and Lakshmanan (RL) studied the solutions of (6) in the form:

$$\phi^{(1)} = \sum_{j=1}^{N} \exp(\xi_j) \qquad \xi_j = k_jx + l_jy - k_j^3t + \xi_j^{(0)}. \tag{8}$$

In order to obtain further information about the dromion solutions of the (2+1)-dimensional KdV equation (1), we can write a special solution of (6) in the following form

$$\phi^{(1)} = \sum_{j=1}^{N}[\exp(k_j(y)x - k_j^3(y)t + g_j(y)) + h_j(x,t)] \tag{9}$$

where $k_j(y)$ and $g_j(y)$ are arbitrary functions of y and $h_j(x,\ t)$ is an arbitrary function of $\{x,\ t\}$. Similarly, the solutions $\phi^{(j)}$ for all $j \geqslant 2$ can be obtained by solving equation (7) etc recursively.

To obtain 'single' soliton solution we select $N = 1$ in (9) and substitute it into equation (7) etc. The result tells us that if we choose k_1 as a constant and h_1 as a solution of the following linear equation

$$h_t + h_{xxx} + 3k_1^2h_x - 3k_1h_{xx} = 0 \tag{10}$$

then $\phi^{(j)}$ for $j \geqslant 2$ can all be selected as zero.

After solving equation (10), we get a generalized real solution of (3):

$$\phi = 1 + \exp\xi + \sum_{j=1}^{M} A_j \exp\eta_j + \sum_{j=1}^{N} B_j \sin\theta_j \exp\zeta_j \tag{11}$$

$$\begin{aligned} &\xi = k_1x - k_1^3t + g(y) \qquad \eta_j = R_jx - (R_j^3 + 3k_1^2R_j - 3k_1R_j^2)t \\ &\theta_j = P_jx + (P_j^3 - 3P_jQ_j^2 - 3k_1^2P_j + 6k_1P_jQ_j)t + \theta_j^{(0)} \\ &\zeta_j = Q_jx + (-Q_j^3 + 3P_j^2Q_j - 3k_1P_j^2 + 3k_1Q_j^2)t \end{aligned} \tag{12}$$

where $k_1,\ A_j,\ R_j,\ P_j,\ Q_j,\ B_j$, and $\theta_j^{(0)}$ are all arbitrary constants and $g(y)$ is an arbitrary function of y. The corresponding forms for the field u and the potential v can be obtained by substituting (11) into (4):

$$\begin{aligned} u = &-2g'(y)\exp\xi\left(k_1 + \sum_{j=1}^{M} A_j(k_1 - R_j)\exp\eta_j + \sum_{j=1}^{N} B_j(k_1 - Q_j)\sin\theta_j\exp\zeta_j\right. \\ &\left.- \sum_{j=1}^{N} B_jP_j\cos\theta_j\exp\zeta_j\right) \\ &\times\left[\left(1 + \exp\xi + \sum_{j=1}^{M} A_j\exp\eta_j + \sum_{j=1}^{N} B_j\sin\theta_j\exp\zeta_j\right)^2\right]^{-1} \end{aligned} \tag{13}$$

$$\begin{aligned} v = &-2\frac{k_1^2\exp\xi + \sum_{j=1}^{M} A_jR_j^2\exp\eta_j + \sum_{j=1}^{N} B_j((Q_j^2 - P_j^2)\sin\theta_j + 2P_jQ_j\cos\theta_j)\exp\zeta_j}{1 + \exp\xi + \sum_{j=1}^{M} A_j\exp\eta_j + \sum_{j=1}^{N} B_j\sin\theta_j\exp\zeta_j} \\ &+2\frac{(k_1\exp\xi + \sum_{j=1}^{M} A_jR_j\exp\eta_j + \sum_{j=1}^{N} B_j(P_j\cos\theta_j + Q_j\sin\theta_j)\exp\zeta_j)^2}{(1 + \exp\xi + \sum_{j=1}^{M} A_j\exp\eta_j + \sum_{j=1}^{N} B_j\sin\theta_j\exp\zeta_j)^2} \end{aligned} \tag{14}$$

where a prime means a derivative with respect to y. To understand the meaning of solutions (13) and (14), we first discuss some special cases:

(i) *Single curve soliton for the potential* v. If we choose

$$A_j = B_j = 0 \tag{15}$$

equation (14) reduces to

$$v = -\tfrac{1}{2}k_1^2 \mathrm{sech}^2 \tfrac{1}{2}(k_1 x - k_1^3 t + g(y)). \tag{16}$$

Obviously, v shown by equation (16) is finite on the curve

$$k_1 x - k_1^3 t + g(y) = 0 \tag{17}$$

and decays exponentially apart from the curve. Hereafter we call this type of soliton a curve soliton (or curved line soliton) and call a solution which is finite on a straight line and decays apart from the line a line (or straight line) soliton.

Under the same condition (15), the structure of soliton (13) for the field u

$$u = -\tfrac{1}{2}k_1 g'(y) \mathrm{sech}^2 \tfrac{1}{2}(k_1 x - k_1^3 t + g(y)) \tag{18}$$

is much more abundant:

(ii) *Single dromion driven by one line soliton (parallel to the x-axis) and one curve soliton.* If $g'(y)$ is fixed as a single line soliton which is parallel to the x-axis, and we combine the line soliton $g'(y)$ and curve soliton $\mathrm{sech}^2(k_1 x - k_1^3 t + g(y))$ together properly (i.e. multiply them together simply in the case of equation (18)), the original straight line *and curved line solitons disappear (become ghosts) and only a single peak which is localized* in all directions (called a dromion) survives. The dromion is located at the intersection of the line and curve solitons. Because $g(y)$ is an arbitrary function of y, the single dromion still possesses rich structures. Here are three concrete simple examples:

$$\begin{aligned} u_1 &= \tfrac{1}{2}k_1 \,\mathrm{sech}^n(y - y_0)\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(y_1 - y_0)\,\mathrm{d}y_1\right) \\ &\equiv \tfrac{1}{2}k_1 h_1(y)\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(y_1 - y_0)\,\mathrm{d}y_1\right) \end{aligned} \tag{19}$$

$$\begin{aligned} u_2 &= \tfrac{1}{2}k_1 \,\mathrm{sech}^n(\cosh(y - y_0) - 1)\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(\cosh(y_1 - y_0) - 1)\,\mathrm{d}y_1\right) \\ &\equiv \tfrac{1}{2}k_1 h_2(y)\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \mathrm{sech}^n(\cosh(y_1 - y_0) - 1)\,\mathrm{d}y_1\right) \end{aligned} \tag{20}$$

$$\begin{aligned} u_3 &= \tfrac{1}{2}k_1 (y - y_0)^{2n} + 1\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \frac{1}{(y_1 - y_0)^{2n} + 1}\,\mathrm{d}y_1\right) \\ &\equiv \tfrac{1}{2}k_1 h_3(y)\,\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \frac{1}{(y_1 - y_0)^{2n} + 1}\,\mathrm{d}y_1\right). \end{aligned} \tag{21}$$

The first type of dromion solution, u_1, decays exponentially in all directions. The second type of dromion solution, u_2, decays much more quickly than the first in the y direction. While the third type of dromion solution, u_3, decays much slower than the first in the y direction.

(iii) *Multi-dromion 'bounded' states.* If $g'(y)$ is selected as N parallel line solitons (parallel to the x-axis) under condition (15), then we obtain an N-dromion 'bound' state,

$$u_{\mathrm{NB}} = \tfrac{1}{2}\left(\sum_{j=1}^{N} f_j(y)\right)\mathrm{sech}^2 \tfrac{1}{2}\left(k_1 x - k_1^3 t - \int^y \sum_{j=1}^{N} f_j(y_1)\,\mathrm{d}y_1\right) \tag{22}$$

driven by N line ghost solitons and one curved line ghost soliton. In equation (22), $f_j(y)$ can be selected quite freely, say h_1, h_2, h_3 shown in equations (19)–(21). Because all N parallel line solitons are static in the y direction, the N-dromions can only move with the same speed in the x direction as the curved line ghost soliton moves and they cannot pass through each other. In other words, the behaviour of these dromions looks like a bound state.

(iv) *Interacting and waving multi-dromions.* If we select $g'(y)$ as an N_1 line soliton solution and remove condition (15), then solution (13) reveals an interacting waving dromion structure. This type of dromion solution is driven by one curve soliton, N_1 line solitons parallel to the x-axis and M non-waving line and N waving line solitons parallel to the y-axis. All the non-waving and waving line solitons which are parallel to the y-axis and the curve soliton move in the x direction at different velocities, while the N_1 line solitons which are parallel to the x-axis are still in the y direction, so only the dromions located at the same y level can interact with each other.

The rich structures of the multi-dromion solution (13) are different from the known traditional dromions such as those in the DS and KP equations. The multi-dromions of the (2+1)-dimensional KdV equation (1) obtained by RL are also different from equation (13). In other words, the multi-dromion solution shown by equation (13) is not included in the multi-dromion solutions obtained by RL [8], neither have the multi-dromion solutions obtained by RL been included in equation (13). To obtain the RL result, we have to consider $\phi^{(1)}$ shown by equation (9) for arbitrary N with k_j being constants:

$$\phi^{(1)} = \sum_{j=1}^{N}[\exp(k_j x - k_j^3 t + g_j(y))] + h(x;t) \tag{23}$$

where $h = \sum_{j=1}^{N} h_j$. Substituting (23) into (7) etc, we find that if we restrict all g_j to have the same form except for some constants:

$$g_j(y) = g_(y) + C_j \qquad (j = 1, 2, \ldots, N,\ C_j = \text{constants}) \tag{24}$$

and $h(x, t)$ is any solution of the linear equation

$$(h_t + h_{xxx}) \sum_{j=1}^{N} \exp(k_j x - k_j^3 t + C_j) + 3 \sum_{j=1}^{N} k_j (k_j h_x - h_{xx}) \exp(k_j x - k_j^3 t + C_j) = 0 \tag{25}$$

$\phi^{(j)}$ for $j \geqslant 2$ can all be treated as zero again. Correspondingly, substituting equation (23) with (24) and (25) into (4), we obtain a more generalized multi-dromion solution.

(v) *Generalized multi-dromion solution.*

$$u = -2g'(y) \frac{(1+h)\sum_{j=1}^{N} k_j \exp \xi_j - h_x \sum_{j=1}^{N} \exp \xi_j}{(1 + \sum_{j=1}^{N} \exp \xi_j + h)^2} \tag{26}$$

where

$$\xi_j = k_j x - k_j^3 t + g(y) + C_j. \tag{27}$$

This type of multi-dromion solution possesses much more abundant structures than that of the first type. In fact, the first type of multi-dromion solution (13) is only a special case of (26) for $N = 1$. From (26) we know that generally, multi-dromions may be driven by many line ghost solitons (both parallel to the x-axis and y-axis) and many curved line ghost solitons.

Now if we take some further restrictions on (26), we would get the dromion solutions studied by RL [8]. For instance, taking

$$N = 2 \qquad k_2 = C_2 = 0 \qquad g(y) = l_1 y + c_2 \qquad C_1 = c_1 + \ln K \tag{28}$$

and

$$h(x,t) = \exp(k_1 x - k_1^3 t + c_1) \tag{29}$$

in (26), the RL (1,1) dromion solution [8]

$$u = \frac{2k_1 l_1 (1-K)\exp(k_1 x + l_1 y - k_1^3 t + c_1 + c_2)}{(1 + \exp(k_1 x - k_1^3 t + c_1) + \exp(l_1 y + c_2) + K\exp(k_1 x + l_1 y - k_1^3 t + c_1 + c_2))^2} \tag{30}$$

follows immediately. Similarly, inserting the special form of (26) for

$$\begin{aligned} &N = 3 \qquad k_3 = C_3 = 0 \qquad g(y) = l_1 y + c_2 \\ &C_1 = c_1 + \ln K \qquad C_2 = c_3 + \ln K \end{aligned} \tag{31}$$

and

$$h(x,t) = \exp(k_1 x - k_1^3 t + c_1) + \exp(k_2 x - k_2^3 t + c_3) \tag{32}$$

into (26) leads to the RL (2, 1) dromion [8].

Finally, we would like to point out that k_j in (23) may be chosen as complex such that the final result $\phi^{(1)}$ is still real thanks to

$$\begin{aligned} &\exp((P + \mathrm{i}Q)x - (P + \mathrm{i}Q)^3 t + g(y)) + \exp((P - \mathrm{i}Q)x - (P - \mathrm{i}Q)^3 t + g(y)) \\ &\quad = 2\cos(Qx - (3P^2Q - Q^3)t)\exp(Px - (P^3 - 3PQ^2)t + g(y)) \end{aligned} \tag{33}$$

being a real function. That is to say, the multi-dromion (26) with (25) can be re-written equivalently as

$$u = -2\partial_x \partial_y \ln \phi \tag{34}$$

$$\phi = 1 + h(x,t) + \sum_{j=1}^{N} \exp \xi_j + \sum_{j=1}^{M} A_j \cos\theta_j \exp \eta_j \tag{35}$$

with

$$\begin{aligned} &\xi_j = k_j x - k_j^3 t + g(y) + C_j, \qquad \theta_j = P_j x - (3Q_j^2 P_j - P_j^3)t + \varphi_j \\ &\eta_j = Q_j x - (Q_j^3 - 3Q_j P_j^2)t + g(y) \end{aligned} \tag{36}$$

and the $h(x,t)$ equation should be modified to

$$\begin{aligned} &(h_t + h_{xxx})\left(\sum_{j=1}^{N} \exp \xi_{1j} + \sum_{j=1}^{M} A_j \cos\theta_j \exp \eta_{1j}\right) + 3\sum_{j=1}^{N}(h_x k_j^2 - h_{xx} k_j)\exp \xi_{1j} \\ &\quad + 3\sum_{j=1}^{M} A_j \left((h_x(Q_j^2 - P_j^2) - h_{xx} Q_j)\cos\theta_j \right. \\ &\quad \left. - (2P_j Q_j h_x - P_j h_{xx})\sin\theta_j\right)\exp \eta_{1j} = 0 \end{aligned} \tag{37}$$

where

$$\xi_{1j} = k_j x - k_j^3 t + C_j \qquad \eta_{1j} = Q_j x - (Q_j^3 - 3Q_j P_j^2)t. \tag{38}$$

In summary, we have obtained many types of new dromion solutions for the (2 + 1)-dimensional KdV equation (1) by solving the general solutions with an arbitrary function for the (2+1)-dimensional bilinear KdV equation. The dromions can be driven not only by some perpendicular line ghost solitons but also by some non-perpendicular line and curved line ghost solitons. The dromions can also possess some quite free shapes. For instance, they may decay extremely rapidly or much slower than the exponentially decayed soliton in the y direction because an arbitrary function is included in the general dromion solution. The dromions may also be waving in the x direction due to some types of exponentially decayed waving line ghost solitons also being included in the general solution. A similar dromion structure with quite a free shape is also found for the breaking soliton equation [10] where the dromions exist for the potential instead of the original physical field [11, 10]. In fact, because all the known high dimensional integrable models possess Kac–Moody–Virasoro type Lie symmetry structures with many arbitrary functions, we firmly believe that some of the properties of dromions revealed here, such as possessing quite free shape in one or more directions and being driven by curved line ghost solitons etc, can exist for all higher dimensional integrable models or their proper potential forms. A study to find dromions driven by curved line ghost solitons for some well known $(2+1)$-dimensional physically significant integrable models such as the DS and KP equations are in progress. It should be noted that the exact closed form for the multi-dromion solution of the $(2+1)$-dimensional KdV equation (1) with more than one arbitrary functions of y has not yet been found and the deeper structures of the dromion solutions obtained here are worthy of further study.

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References

[1] Ablowitz M, Fuchssteiner B and Kruskal M 1987 *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations* (Singapore: World Scientific)
Fokas A S and Santini 1989 *Phys. Rev. Lett.* **63** 1329
[2] Boiti M, Leon J J P, Martina L and Penpinelli F 1988 *Phys. Lett.* **132A** 432
Fokas A S and Santini P M 1990 *Physica* D **44** 99
[3] Hietarinta J 1990 *Phys. Lett.* **149A** 133
[4] Winternitz P 1993 Lie group and solutions of nonlinear partial differential equations *Preprint* CRM 1841 and references therein
Tamizhmani K M, Ramani R and Gramaticos B 1991 *J. Math. Phys.* **32** 2635
[5] Lou S-y 1993 *Phys. Rev. Lett.* **71** 4099; *J. Phys. A: Math. Gen.* **26** 4387; 1994 *J. Phys. A: Math. Gen.* **27** 3235; L207; *J. Math. Phys.* 1755
[6] Lou S-y, Yu J and Lin J 1995 *J. Phys. A: Math. Gen.* **28** L19
Lou S-y, Lin J and Yu J 1995 *Phys. Lett.* **201A** 47
[7] Boiti M, Leon J J P, Manna M and Penpinelli F 1986 *Inverse Problems* **2** 271; 1987 *Inverse Problems* **3** 25
[8] Radha R and Lakshmanan M 1994 *J. Math. Phys.* **35** 4746
[9] Hirota R 1971 *Phys. Rev. Lett.* **27** 1192
[10] Lou S-y 1995 *Preprint* NBN-IMP 08/95.
[11] Radha R and Lakshmanan M 1995 *Phys. Lett.* **197A** 7